

MODULI SPACES OF VECTOR BUNDLES ON A SINGULAR RATIONAL RULED SURFACE

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ABSTRACT. We study moduli spaces $M_X(r, c_1, c_2)$ parametrizing slope semistable vector bundles of rank r and fixed Chern classes c_1, c_2 on a ruled surface whose base is a rational nodal curve. We show that under certain conditions, these moduli spaces are irreducible, smooth and rational (when non-empty). We also prove that they are non-empty in some cases.

We show that for a rational ruled surface defined over real numbers, the moduli space $M_X(r, c_1, c_2)$ is rational as a variety defined over \mathbb{R} .

1. INTRODUCTION

Vector bundles on smooth complex ruled surfaces have been studied by many authors from different points of view, the case of rank two being studied most extensively. Let X be a complex projective surface equipped with a polarization H , and let $M_{X,H}(r, c_1, c_2)$ denote the moduli space of H -semistable (slope semistable) vector bundles on X of rank r with fixed Chern classes $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbb{Z}$. When X is a smooth ruled surface or a blow up of it, Walter, [Wa], found a precise sufficient condition on H for $M_{X,H}(r, c_1, c_2)$ to be irreducible whenever it is non-empty. He also proved that this moduli space is normal and its subvariety $M_{X,H}^s(r, c_1, c_2)$ that parametrizes stable vector bundles is smooth. Furthermore, he gave examples of X and H (not satisfying his condition) for which the moduli spaces $M_{X,H}(2, c_1, c_2)$ are reducible for some small c_1, c_2 .

Another interesting property investigated by many is the rationality of the scheme $M_{X,H}(r, c_1, c_2)$. The question is the following: If X is rational, is $M_{X,H}(r, c_1, c_2)$ also rational? Although several cases are known where the answer is positive, the answer is not known in general. Costa and Miró-Roig explicitly constructed generic H -stable vector bundles on a smooth Hirzebruch surface X for many values of r, c_1, c_2 , [CM], and showed non-emptiness for these moduli spaces. They also proved that the moduli space is a rational variety in these cases [CM, Theorem A].

In this paper, we generalize these results to singular rational ruled surfaces. Let X be a complex rational ruled surface whose base is an integral rational projective curve Y of arithmetic genus g with g nodes (ordinary double points) as its only singularities. We study the geometric properties of X . In particular, we prove that X is a Gorenstein variety, compute its invariants and determine the dualizing sheaf explicitly. Following

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[Wa], we establish a sufficient condition on H for the moduli space $M_{X,H}(r, c_1, c_2)$ to be irreducible. Furthermore, we prove the existence of polarizations H satisfying this condition (see Theorem 3.7).

We also investigate the rationality question for $M_{X,H}(r, c_1, c_2)$. Let $\pi: \bar{Y} \rightarrow Y$ be the normalization map for the base curve. If $X := \mathbb{P}(\mathcal{E})$, then $Z := \mathbb{P}(\pi^*\mathcal{E})$ is a smooth Hirzebruch surface. Let H_Z denote the polarization on Z which is the pullback of the polarization H on X . We show that whenever $M_{Z,H_Z}^s(r, c_1, c_2)$ is rational, the variety $M_{X,H}^s(r, c_1, c_2)$ is also rational (see Theorem 4.3). In view of the results of [CM], this yields rationality of $M_{X,H}^s(r, c_1, c_2)$ in several cases.

Finally we study singular real rational ruled surfaces $X_{\mathbb{R}}$ whose base $Y_{\mathbb{R}}$ a rational curve defined over \mathbb{R} . Let $\pi': C \rightarrow Y_{\mathbb{R}}$ be the normalization map. Let $\mathcal{E}_{\mathbb{R}}$ be a real vector bundle of rank 2 over $Y_{\mathbb{R}}$ such that $X_{\mathbb{R}} = \mathbb{P}(\mathcal{E}_{\mathbb{R}})$. Let $Z_{\mathbb{R}} := \mathbb{P}(\pi'^*\mathcal{E}_{\mathbb{R}})$ be the real ruled surface with base C . For a real ruled surface $Z_{\mathbb{R}}$ with base an anisotropic conic C , the rationality question of $M_{Z_{\mathbb{R}},H_Z}^s(r, c_1, c_2)$ was studied in [BS].

We prove that $M_{X_{\mathbb{R}},H}^s(r, c_1, c_2)$ is a real rational variety if $M_{Z_{\mathbb{R}},H_Z}^s(r, c_1, c_2)$ is a real rational variety (see Theorem 5.2). Coupled with the results of [BS], this gives necessary and sufficient conditions for $M_{X_{\mathbb{R}},H}^s(r, c_1, c_2)$ to be a real rational variety (Theorem 5.3).

2. SINGULAR RULED SURFACES

In this section, we define a singular rational ruled surface that we are interested in and study its properties.

2.1. Notation. Let Y be an integral rational projective curve of arithmetic genus g over \mathbb{C} with only nodes (ordinary double points) as singularities. Therefore, Y has exactly g singular points. Let y_1, \dots, y_g be the singular points of Y . Let

$$\pi: \bar{Y} \rightarrow Y$$

be the normalization map. Then $\bar{Y} = \mathbb{P}_{\mathbb{C}}^1$ because Y is rational. For $1 \leq j \leq g$, let $\{x_j, z_j\} \subset \bar{Y}$ be the pair of points over $y_j \in Y$.

Take an algebraic vector bundle \mathcal{E} over Y of rank two and degree $-e$. Let

$$X := \mathbb{P}(\mathcal{E}) : X \xrightarrow{p_1} Y$$

be the corresponding $\mathbb{P}_{\mathbb{C}}^1$ -bundle over Y . Then

$$Z := \mathbb{P}(\mathcal{E}) \times_Y \bar{Y} = \mathbb{P}(\pi^*\mathcal{E})$$

is a smooth Hirzebruch surface. Let

$$p_0: Z \rightarrow \bar{Y}$$

be the projection to the second factor of the fiber product. The relatively ample tautological line bundles on X and Z will be denoted by $\mathcal{O}_{p_1}(1)$ and $\mathcal{O}_{p_0}(1)$ respectively.

We fix an ample line bundle H on X . Let

$$p: Z = X \times_Y \bar{Y} \rightarrow X$$

be the projection to the first factor of the fiber product. Define the line bundle

$$H_Z := p^*H \longrightarrow Z.$$

Since p is a finite map, and H is ample, it follows that H_Z is ample.

By tensoring \mathcal{E} with a line bundle, we may assume that $Z = \mathbb{P}(\mathcal{O}_{\overline{Y}} \oplus \mathcal{L})$, because on one hand $\pi^*\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2 \cong \mathcal{L}_1 \otimes (\mathcal{O}_{\overline{Y}} \oplus (\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}))$, on the other hand, $\mathcal{L}_1 = \pi^*N$ for some line bundle N on Y , so $\pi^*(\mathcal{E} \otimes N^{-1}) \cong \mathcal{O}_{\overline{Y}} \oplus (\mathcal{L}_2 \otimes \mathcal{L}_1^{-1})$. The inclusion of $\mathcal{O}_{\overline{Y}}$ in $\mathcal{O}_{\overline{Y}} \oplus (\mathcal{L}_2 \otimes \mathcal{L}_1^{-1})$ defines an irreducible effective divisor on Z ; we denote this divisor by C_0 .

We have a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ p_0 \downarrow & & p_1 \downarrow \\ \overline{Y} & \xrightarrow{\pi} & Y \end{array}$$

For $1 \leq j \leq g$, define

$$P_j := p_1^{-1}(y_j), \quad P_{x_j} := p_0^{-1}(x_j), \quad P_{z_j} := p_0^{-1}(z_j),$$

so $p^{-1}(P_j) = P_{x_j} \amalg P_{z_j}$. The restrictions $p|_{P_{x_j}}$ and $p|_{P_{z_j}}$ identify P_j with P_{x_j} and P_{z_j} respectively. Therefore, we obtain a canonical isomorphism

$$\tau_j : P_{x_j} \xrightarrow{\sim} P_{z_j}. \quad (2.1)$$

We note that τ_j is induced by the canonical identification of $(\pi^*\mathcal{E})_{x_j}$ with $(\pi^*\mathcal{E})_{z_j}$.

The Hirzebruch surface Z has been studied extensively (see [Ha2, Chapter V] for generalities on Hirzebruch surfaces). We start with some geometric properties of X .

Lemma 2.1.

- (1) *The variety X is semi-normal; the disjoint union $\bigcup_{j=1}^g P_j$ is the non-normal locus.*
- (2) *The variety X is Gorenstein.*
- (3) *The dualizing sheaf ω_X of X is a locally free sheaf of rank one.*

Proof. (1): As the singularities of Y are ordinary nodes, Y is a semi-normal variety. Locally, X is a product of a semi-normal variety with a normal (in fact non-singular) variety and hence X is a semi-normal variety [GT, Proof of Corollary 5.9].

Since $p^{-1}(Y - \bigcup_{j=1}^g y_j)$ is non-singular, the last assertion in (1) follows.

(2): The fibers of p_1 are non-singular and hence Gorenstein. The morphism p_1 is flat, the base Y is Gorenstein, and the fibers of p_1 are also Gorenstein. Therefore, it follows that X is Gorenstein [Ha1, Proposition 9.6].

(3): Since X is Gorenstein (by part (2)), the dualizing sheaf ω_X is a locally free sheaf of rank 1 [Ha1, p. 295–296, Theorem 9.1]. \square

The following lemma, which sums up facts about X , is an easy but a useful one.

Lemma 2.2.

- (1) *The Picard group $\text{Pic}(X) = p_1^*\text{Pic}(Y) \oplus \mathbb{Z}\mathcal{O}_{p_1}(1)$.*

(2) *Invariants of X :*

*Arithmetic genus $p_a(X) := \chi(\mathcal{O}_X) - 1 = -g$,
 geometric genus $p_g(X) := H^2(X, \mathcal{O}_X) = 0$, and
 irregularity $q(X) := H^1(X, \mathcal{O}_X) = g$.*

(3) $h^0(X, \omega_X) = 0$, $h^1(X, \omega_X) = g$ and $h^2(X, \omega_X) = 1$.

Proof. Since $p_1 : X \rightarrow Y$ is a \mathbb{P}^1 -bundle, the first statement follows.

Since $(p_1)_*\mathcal{O}_X = \mathcal{O}_Y$, we have $h^i(X, \mathcal{O}_X) = h^i(Y, \mathcal{O}_Y) \forall i$, hence $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$. Therefore (2) follows.

Statement (3) follows from (2) and Serre duality. \square

Remark 2.3. For any $y \in Y$, the fiber $p_1^{-1}(y)$ is isomorphic to \mathbb{P}^1 . However, if y is a non-singular point, then the fiber is a Cartier divisor, otherwise it is not a Cartier divisor. For y non-singular, the Cartier divisor

$$F_y := p_1^{-1}(y)$$

corresponds to the line bundle $p_1^*\mathcal{O}_Y(y)$. For a node $y = y_j$, the fiber F_{y_j} is locally defined by two equations. The ideal sheaf $I(F_{y_j})$ of F_{y_j} in \mathcal{O}_X is isomorphic to $p_1^*I(y_j)$, where $I(y_j)$ denotes the ideal sheaf of y_j in \mathcal{O}_Y .

Proposition 2.4. *The dualizing sheaf ω_X is isomorphic to $p_1^*(\omega_Y \otimes \det \mathcal{E}) \otimes \mathcal{O}_{p_1}(-2)$.*

Proof. Since \overline{Y} and Y are Gorenstein curves, and π is a finite map, we conclude that $\omega_{\overline{Y}} \cong \pi^*\omega_Y \otimes \mathcal{C}_{\overline{Y}/Y}$. As the conductor sheaf $\mathcal{C}_{\overline{Y}/Y}$ is isomorphic to $\mathcal{O}_{\overline{Y}}(-\sum_j(x_j + z_j))$, we have

$$\omega_{\overline{Y}} \otimes \mathcal{O}_{\overline{Y}}(\sum_j(x_j + z_j)) \cong \pi^*\omega_Y.$$

Similarly, since X and Z are both Gorenstein varieties (see Lemma 2.1), and p is a finite map between them, one has

$$\omega_Z \cong p^*\omega_X \otimes \mathcal{C}_{Z/X},$$

where $\mathcal{C}_{Z/X}$ is the conductor sheaf. We have $\mathcal{C}_{Z/X} \cong p_0^*\mathcal{C}_{\overline{Y}/Y}$. Therefore, it follows that $\mathcal{C}_{Z/X} \cong \mathcal{O}_Z(-\sum_j(P_{x_j} + P_{z_j}))$. Hence

$$p^*\omega_X \cong \omega_Z \otimes \mathcal{O}_Z(\sum_j(P_{x_j} + P_{z_j})).$$

It is known that $\omega_Z \cong p_0^*(\omega_{\overline{Y}} \otimes (\det \pi^*\mathcal{E})) \otimes \mathcal{O}_{p_0}(-2)$ [Ha2, Chapter V, Lemma 2.10]. Hence

$$\begin{aligned} p^*\omega_X &\cong p_0^*(\omega_{\overline{Y}} \otimes (\det \pi^*\mathcal{E}) \otimes \mathcal{O}_{\overline{Y}}(\sum_j(x_j + z_j))) \otimes \mathcal{O}_{p_0}(-2) \\ &\cong p_0^*(\pi^*\omega_Y \otimes (\det \pi^*\mathcal{E})) \otimes \mathcal{O}_{p_0}(-2) \\ &\cong p_0^*(\pi^*(\omega_Y \otimes (\det \mathcal{E}))) \otimes p^*\mathcal{O}_{p_1}(-2) \\ &\cong p^*(p_1^*(\omega_Y \otimes (\det \mathcal{E}))) \otimes p^*\mathcal{O}_{p_1}(-2) \\ &\cong p^*(p_1^*(\omega_Y \otimes (\det \mathcal{E})) \otimes \mathcal{O}_{p_1}(-2)). \end{aligned}$$

Thus

$$p^*\omega_X \cong p^*(p_1^*(\omega_Y \otimes (\det \mathcal{E})) \otimes \mathcal{O}_{p_1}(-2)).$$

This, combined with the facts that

$$\mathrm{Pic}(X) \cong p_1^* \mathrm{Pic}(Y) \oplus \mathbb{Z} \mathcal{O}_{p_1}(1), \quad \mathrm{Pic}(Z) \cong p_0^* \mathrm{Pic}(\overline{Y}) \oplus \mathbb{Z} \mathcal{O}_{p_0}(1)$$

(see Lemma 2.2(1)) and $p^* \mathcal{O}_{p_1}(1) \cong \mathcal{O}_{p_0}(1)$, implies that

$$\omega_X \cong p_1^*(\omega_Y \otimes (\det \mathcal{E}) \otimes N) \otimes \mathcal{O}_{p_1}(-2),$$

for some line bundle N on Y whose pull-back to \overline{Y} is trivial.

Taking direct image on Y , we have

$$R^i(p_1)_* \omega_X \cong (\omega_Y \otimes (\det \mathcal{E}) \otimes N) \otimes R^i(p_1)_* \mathcal{O}_{p_1}(-2).$$

Since $H^i(\mathbb{P}^1, \mathcal{O}(-2)) = 0 \ \forall i \neq 1$, and $H^1(\mathbb{P}^1, \mathcal{O}(-2)) = 1$, one has

$$R^i(p_1)_* \mathcal{O}_{p_1}(-2) = 0 \ \forall i \neq 1$$

and $R^1(p_1)_* \mathcal{O}_{p_1}(-2)$ is a line bundle. Tensoring the Euler sequence

$$0 \longrightarrow (\det p_1^* \mathcal{E}) \otimes \mathcal{O}_{p_1}(-1) \longrightarrow p_1^* \mathcal{E} \longrightarrow \mathcal{O}_{p_1}(1) \longrightarrow 0$$

with $\mathcal{O}_{p_1}(-1)$ and taking direct image by p_1 , one gets $R^1(p_1)_* \mathcal{O}_{p_1}(-2) \cong \det \mathcal{E}^*$. Hence we have

$$(p_1)_* \omega_X = 0, \quad R^2(p_1)_* \omega_X = 0$$

and the only non-vanishing direct image is $R^1(p_1)_* \omega_X \cong \omega_Y \otimes N$. Consequently, we have

$$h^1(X, \omega_X) = h^0(Y, R^1 p_{1*} \omega_X) = h^0(Y, \omega_Y \otimes N).$$

Now Lemma 2.2(3) implies that $h^0(Y, \omega_Y \otimes N) = g$. By Serre duality, $h^1(Y, N^*) = g$. So by Riemann–Roch, we have $h^0(Y, N^*) = 1$. Since $d(N^*) = 0$, from $h^0(Y, N^*) = 1$ it follows that N^* , and hence N , is the trivial line bundle. This completes the proof of the proposition. \square

3. IRREDUCIBILITY OF THE MODULI SPACE $M(r, c_1, c_2)$.

Our goal in this section is to prove that the moduli scheme $M_{X,H}(r, c_1, c_2)$ of H -stable vector bundles on X of rank r and with fixed Chern classes c_1, c_2 is irreducible if H satisfies suitable conditions. We closely follow the proof in [Wa] where the irreducibility of $M_{X,H}(r, c_1, c_2)$ is proved under the assumption that X is a smooth Hirzebruch surface. Hence we mainly explain the line of proof and the modifications needed to cover the singular case. Some details are omitted citing appropriate references to [Wa].

Definition 3.1. A coherent sheaf E on X is called *prioritary* (with respect to p_1) if it is torsionfree and $\mathrm{Ext}^2(E, E(-F_y)) = 0$, where F_y denotes the Cartier divisor defined in Remark 2.3.

In the stack of coherent sheaves on X , the *prioritary* sheaves on X are parametrized by an open substack (by semicontinuity theorem). Let

$$\mathrm{Prior}_X(r, c_1, c_2) \subset \mathrm{Coh}_X(r, c_1, c_2)$$

denote the stack of priority sheaves on X of rank r and with fixed Chern classes c_1, c_2 . Let

$$\mathrm{H}\text{-SS}_X^{\mathrm{vect}}(r, c_1, c_2) \subset \mathrm{Prior}_X(r, c_1, c_2)$$

denote the substack of H -semistable priority vector bundles on X of rank r and Chern classes c_1, c_2 .

For convenience (by abuse of notation), we denote $c_1(\omega_X \otimes \mathcal{O}_X(F_y))$ by $\omega_X + F_y$ and $c_1(H)$ by H again. Then the intersection (or cup product), of $c_1(\omega_X \otimes \mathcal{O}_X(F_y))$ with $c_1(H)$, evaluated on the fundamental cycle (or cap product with fundamental class) $[X]$ will be denoted by $H \cdot (\omega_X + F_y)$. With these notations, we have the following lemma.

Lemma 3.2. *If $H \cdot (\omega_X + F_y) < 0$ for a general fiber F_y (Remark 2.3), then any H -semistable sheaf E is priority.*

Proof. Suppose that $\mathrm{Ext}^2(E, E(-F_y)) \neq 0$ for some non-singular point $y \in Y$. Then by Serre duality, there exists a non-zero element $\phi \in \mathrm{Hom}(E, E(\omega_X + F_y))$. Let $\mathrm{Im}(\phi)$ denote the image of the homomorphism ϕ . By the H -semistability of E and $E(\omega_X + F_y)$, we get

$$\mu_H(E) \leq \mu_H(\mathrm{Im}(\phi)) \leq \mu_H(E(\omega_X + F_y)) = \mu_H(E) + H \cdot (\omega_X + F_y). \quad (3.1)$$

Since $H \cdot (\omega_X + F_y) < 0$, this gives a contradiction. Thus E is priority.

We note that the last equality in (3.1) may not hold for a singular point y , hence the proof fails if F_y is not a general fiber. \square

Lemma 3.3. *The stack $\mathrm{H}\text{-SS}_X^{\mathrm{vect}}(r, c_1, c_2)$ is smooth.*

Proof. It suffices to show that $\mathrm{Ext}^2(E, E) = 0$ for an H -semistable vector bundle E . Let y be a non-singular point of Y . One has a short exact sequence

$$0 \longrightarrow E(-F_y) \longrightarrow E \longrightarrow E|_{F_y} \longrightarrow 0.$$

Applying $\mathrm{Hom}(E, -)$ to this exact sequence yields

$$\longrightarrow \mathrm{Ext}^2(E, E(-F_y)) \longrightarrow \mathrm{Ext}^2(E, E) \longrightarrow \mathrm{Ext}^2(E, E|_{F_y}) \longrightarrow .$$

We have

$$\mathrm{Ext}^2(E, E|_{F_y}) \cong H^2(F_y, E^* \otimes E|_{F_y}) = 0$$

as $F_y \cong \mathbb{P}^1$, and hence $\mathrm{Ext}^2(E, E|_{F_y}) = 0$. Since E is priority, $\mathrm{Ext}^2(E, E(-F_y)) = 0$. It follows that $\mathrm{Ext}^2(E, E) = 0$. \square

Lemma 3.4. *Let σ be a section of $p_1 : X \longrightarrow Y$. Let E be a coherent sheaf on X such that $(p_1)_*(E(-\sigma)) = 0 = R^1(p_1)_*E$. Then there is an exact sequence*

$$0 \longrightarrow p_1^*(p_{1*}E) \longrightarrow E \longrightarrow p_1^*(R^1p_{1*}(E(-\sigma)) \otimes \Omega_{X/Y}(\sigma)) \longrightarrow 0.$$

Proof. This is essentially [Wa, Lemma 8]. The proof goes through in the singular case as [Be, Remark 3] implies that the Beilinson resolution exists over any base Y . \square

Proposition 3.5. *The stack $\mathrm{Prior}_X(r, c_1, c_2)$ of priority sheaves is irreducible.*

Proof. We briefly sketch a proof (see [Wa, Proposition 2] for details). Fix a section $\sigma \in X$. Let $d = -c_1 \cdot F_y$. We may assume that $0 \leq d < r$ (by twisting E with a power of $\mathcal{O}_X(\sigma)$). If $d > 0$, we may restrict ourselves to the dense open substack $Prior^0 \subset Prior_X(r, c_1, c_2)$ parametrizing all E such that $E|_{F_y} \cong \mathcal{O}_{F_y}^{r-d} \oplus (\mathcal{O}_{F_y}(-1))^d$ (since the complement forms a closed substack of codimension at least one). If $d = 0$, we may restrict ourselves to the dense open substack $Prior^0 \subset Prior_X(r, c_1, c_2)$ parametrizing all E such that $E|_{F_y} \cong \mathcal{O}_{F_y}^r$ for all but finitely many $y \in Y$ and $E|_{F_y} \cong \mathcal{O}_{F_y}(1) \oplus \mathcal{O}_{F_y}^{r-2} \oplus (\mathcal{O}_{F_y}(-1))$ at these finitely many points.

In either case, one sees that $K = p_{1*}E$ is a vector bundle on Y of rank $r - d$ and degree $k = \chi(E) + (r - d)(g - 1)$. Also $L = R^1 p_{1*}(E(-\sigma))$ is a sheaf of rank d , degree $\ell = -\chi(E) + (c_1 \cdot \sigma) - (r - d)(g - 1)$. Moreover, L is locally free for $d > 0$ and a skyscraper sheaf for $d = 0$.

By Lemma 3.4, there exists a short exact sequence

$$0 \longrightarrow p_1^* K \longrightarrow E \longrightarrow p_1^* L \otimes \Omega_{X/Y}(\sigma) \longrightarrow 0.$$

By [DL, p. 200], if E has a filtration

$$\mathcal{F} : 0 = F_0 \subset F_1 \subset \cdots \subset F_t = E, \quad gr_i(E) := (E_i/E_{i-1}),$$

one defines groups $\text{Ext}_+^i(E, E)$ and $\text{Ext}_-^i(E, E)$ such that there is an exact sequence

$$\cdots \longrightarrow \text{Ext}_-^i(E, E) \longrightarrow \text{Ext}^i(E, E) \longrightarrow \text{Ext}_+^i(E, E) \longrightarrow \cdots.$$

Then there is a spectral sequence for which

$$E_1^{p,q} = \prod_i \text{Ext}^{p+q}(gr_i(E), gr_{i-p}(E)) \text{ if } p < 0; \quad E_1^{p,q} = 0 \text{ for } p \geq 0,$$

and which converges to $\text{Ext}_+^\bullet(E, E)$.

In our case, for $\mathcal{F} : 0 \subset \pi^* K \subset E$, we have

$$\text{Ext}_+^i(E, E) = H^i(p_1^*(K^* \otimes L) \otimes \Omega_{X/Y}(\sigma)) = H^i(K^* \otimes L \otimes p_{1*} \mathcal{O}_{p_1}(-1)) = 0 \quad \forall i.$$

Hence $\text{Ext}_-^i(E, E) = \text{Ext}^i(E, E)$ for all i , so that infinitesimal deformations of E are same as those of $0 \subset \pi^* K \subset E$.

By [DL, Remark on p. 201], there is a spectral sequence with

$$E_1^{p,q} = \prod_i \text{Ext}^{p+q}(gr_i(E), gr_{i-p}(E)) \text{ if } p \geq 0; \quad E_1^{p,q} = 0 \text{ for } p < 0,$$

and which converges to $\text{Ext}_-^\bullet(E, E)$.

Since

$$\text{Ext}^2(p_1^* L \otimes \Omega_{X/Y}(\sigma), p_1^* K) = 0,$$

straightforward computations show that

$$\text{Ext}^1(E, E) = \text{Ext}_-^1(E, E)$$

surjects onto $\text{Ext}^1(K, K) \oplus \text{Ext}^1(L, L)$. Hence a general infinitesimal deformation of E induces a general infinitesimal deformation of K and L , and the morphism

$$\phi : Prior^0 \longrightarrow Coh_Y(r - d, k) \times Coh_Y(d, \ell), \quad [E] \longmapsto ([K], [L])$$

is dominant. Since the stack of vector bundles of fixed rank and degree on a nodal curve is irreducible, [Re], and every coherent sheaf is in the limit of vector bundles, the stack of coherent sheaves of fixed rank and degree on a nodal curve is irreducible. Hence the image of ϕ is irreducible. The fibers of ϕ are stack quotients of an affine subscheme of the affine space $\text{Ext}^1(p_1^*L \otimes \Omega_{X/Y}(\sigma), p_1^*K)$, hence they are irreducible. It follows that $\text{Prior}_X(r, c_1, c_2)$ is irreducible. \square

Lemma 3.3 and Proposition 3.5 together give the following:

Corollary 3.6. *The substack $\text{H-SS}_X^{\text{vect}}(r, c_1, c_2)$ is a smooth irreducible open substack of the stack $\text{Prior}_X(r, c_1, c_2)$.*

Theorem 3.7. *The moduli space $M_{X,H}(r, c_1, c_2)$ of H -semistable vector bundles on X is a normal irreducible variety. Its open subscheme corresponding to stable vector bundles is a smooth variety.*

Proof. This can be proved on similar lines as the corresponding part of the proof of [Wa, p. 208, Theorem 1]. The smooth irreducible stack $\text{H-SS}_X^{\text{vect}}$ (see Corollary 3.6) is a quotient stack $[Q^{ss}/\text{GL}(N)]$, where Q^{ss} is an open subscheme of a quot scheme. Hence Q^{ss} is a smooth irreducible scheme. The moduli space $M_{X,H}(r, c_1, c_2)$ is the GIT quotient $Q^{ss}/\text{PGL}(N)$ for Simpson's polarization on Q^{ss} (see [Si]). Hence $M_{X,H}(r, c_1, c_2)$ is a normal irreducible variety and its open subscheme corresponding to stable points, i.e., the open subscheme corresponding to stable vector bundles, is a smooth variety. \square

Lemma 3.8. *There exists an ample line bundle H on X such that*

$$H \cdot \omega_X + F_y < 0.$$

Proof. This follows essentially imitating the proof of the first part of [Wa, Lemma 10]. Let the notations be as in Lemma 3.2. By Lemma 2.2, we may take $H = bF_y + \sigma$, $b \gg 0$, where σ is the class of $\mathcal{O}_{p_1}(1)$. One has

$$F_y \cdot F_y = 0, \quad F_y \cdot \sigma = 1 \quad \text{and} \quad \sigma \cdot \sigma = \mathcal{O}_{p_0}(1) \cdot \mathcal{O}_{p_0}(1) = -e$$

by [Ha2, Chapter V, Proposition 2.9]. Using Proposition 2.4 it follows that the class of $\omega_X + F_y$ is $(2g - 1 - e)F_y - 2\sigma$. Hence

$$\begin{aligned} H \cdot (\omega_X + F_y) &= (bF_y + \sigma)((2g - 1 - e)F_y - 2\sigma) \\ &= -2b + 2g - 1 - e + 2e \\ &= -2b + 2g - 1 - e. \end{aligned}$$

It follows that $H \cdot (\omega_X + F_y) < 0$ for $b \gg 0$, i.e., $(c_1(H) \cup c_1(\omega_X + \mathcal{O}_X(F_y))) \cap [X] < 0$ for $b \gg 0$. \square

4. RATIONALITY OF $M_X(r, c_1, c_2)$

For a coherent sheaf W on projective variety \mathcal{M} equipped with a very ample line bundle \mathcal{H} , define $W(m) := W \otimes \mathcal{H}^{\otimes m}$. Let $P_{\mathcal{M}}(W, m)$ denote the Hilbert polynomial of W with

respect to \mathcal{H} [HL]. Then

$$P_{\mathcal{M}}(W, m) = \sum_{i=0}^{\dim \text{supp.}(W)} a_i(W) \frac{m^i}{i!}, \quad a_i(W) \text{ integers.}$$

The rank of W is $r(W) = \frac{a_d(W)}{a_d(\mathcal{O}_{\mathcal{M}})}$, $d = \dim \mathcal{M}$, and the degree of W is $d(W) = a_{d-1}(W) - r(W)a_{d-1}(\mathcal{O}_{\mathcal{M}})$.

By the projection formula,

$$(p_*E)(m) := p_*E \otimes H^{\otimes m} \cong p_*(E \otimes p^*H^{\otimes m}) = p_*(E \otimes H_Z^{\otimes m}) = p_*(E(m)).$$

Hence $H^i(X, (p_*E)(m)) = H^i(X, p_*(E(m))), \forall i$. Since p is a finite map,

$$H^i(X, p_*(E(m))) = H^i(Z, E(m)).$$

It follows that

$$P_X(p_*E, m) = P_Z(E, m). \quad (4.1)$$

One has a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow p_*\mathcal{O}_Z \longrightarrow \bigoplus_{j=1}^g \mathcal{O}_{P_j} \longrightarrow 0. \quad (4.2)$$

Tensoring (4.2) with $H^{\otimes m}$ and using (4.1), we see that the Hilbert polynomials satisfy the equation

$$P_X(\mathcal{O}_X, m) = P_Z(\mathcal{O}_Z, m) + \sum_j \chi(\mathcal{O}_{P_j}(m)).$$

Since $d(H|_{P_j}) = \ell$, we have $\chi(\mathcal{O}_{P_j}(m)) = \ell m + 1$. Hence comparing the coefficients of powers of m it follows that

$$a_2(\mathcal{O}_X) = a_2(\mathcal{O}_Z), \quad a_1(\mathcal{O}_X) = a_1(\mathcal{O}_Z) + \ell g, \quad a_0(\mathcal{O}_X) = a_0(\mathcal{O}_Z) + g. \quad (4.3)$$

4.1. Generalized parabolic bundles. Here we give a definition of a generalized parabolic bundle (GPB for short) suitable in our special case. For a more general definition of a GPB and generalities on them the reader may refer to [Bh1, Section 2].

Let F be a vector bundle on X and $E := p^*F$ its pullback to Z . Since $E|_{P_{x_j}} \cong p^*(F|_{P_j}) \cong E|_{P_{z_j}}$, we get a canonical isomorphism

$$\sigma_j : E|_{P_{x_j}} \xrightarrow{\sim} E|_{P_{z_j}}$$

lying over the isomorphism τ_j in (2.1). Let $\sigma := (\sigma_1, \dots, \sigma_g)$. Thus a vector bundle F on X determines a pair (E, σ) as above. We call such a pair a generalized parabolic bundle.

Conversely, given a GPB (E, σ) on Z , we get a vector bundle F on X in the following way. Let $F_j(E) \subset E|_{P_{x_j}} \oplus E|_{P_{z_j}}$ denote the graph of σ_j . The surjective morphism $E \longrightarrow E|_{P_{x_j}} \oplus E|_{P_{z_j}}$ produces a surjection of \mathcal{O}_X -modules

$$p_*E \longrightarrow \bigoplus_j p_*((E|_{P_{x_j}} \oplus E|_{P_{z_j}})/F_j(E)).$$

Let F be the kernel of the latter surjection, so F fits in the exact sequence

$$0 \longrightarrow F \longrightarrow p_*E \longrightarrow \bigoplus_j p_*((E|_{P_{x_j}} \bigoplus E|_{P_{z_j}})/F_j(E)) \longrightarrow 0.$$

One sees that $p^*F = E$, and hence E and F have the same rank and same Chern classes. The above construction gives a bijective correspondence between GPBs of rank r and Chern classes c_1, c_2 on Z and vector bundles of rank r and Chern classes c_1, c_2 on X .

Lemma 4.1. *Let (E, h) be a GPB on Z determining a vector bundle F on X . Let $E_1 \subset E$ be a torsion free subsheaf such that E/E_1 is torsionfree. Then E_1 determines a subsheaf $F_1 \subset F$ such that $\mu(F_1) \leq \mu(E_1)$.*

Proof. Since the quotient E/E_1 is torsion free, it follows that

$$E_1|_{P_{x_j}} \oplus E_1|_{P_{z_j}} \subset E|_{P_{x_j}} \oplus E|_{P_{z_j}}.$$

Let

$$F_j(E_1) := F_j(E) \cap (E_1|_{P_{x_j}} \oplus E_1|_{P_{z_j}}), \quad Q_j(E_1) := (E_1|_{P_{x_j}} \oplus E_1|_{P_{z_j}})/F_j(E_1).$$

Define the sheaf F_1 on X by the following short exact sequence

$$0 \longrightarrow F_1 \longrightarrow p_*E_1 \longrightarrow \bigoplus_j p_*Q_j(E_1) \longrightarrow 0. \quad (4.4)$$

Since h is an isomorphism, the projection $pr_j : F_j(E) \rightarrow E|_{P_{x_j}}$ is an isomorphism. Therefore,

$$pr_j|_{F_j(E_1)} \rightarrow E_1|_{P_{x_j}}$$

is an injection. Hence $r(F_j(E_1)) \leq r(E_1|_{P_{x_j}})$. Similarly, we have $r(F_j(E_1)) \leq r(E_1|_{P_{z_j}})$. Hence tensoring (4.4) by $H^{\otimes m}$, we have

$$0 \longrightarrow F_1(m) \longrightarrow (p_*E_1)(m) \longrightarrow \bigoplus_j p_*Q_j(E_1)(m) \longrightarrow 0. \quad (4.5)$$

Therefore, $P_X(F_1, m) = P_X(p_*E_1, m) - \sum_j P_{P_j}(p_*(Q_j(E_1), m))$. By (4.1),

$$P_X(p_*E_1, m) = P_Z(E_1, m) \quad \text{and} \quad P_{P_j}(p_*Q_j(E_1), m) = \chi_{(P_j)}(p_*Q_j(E_1), m).$$

Hence comparing coefficients of m in (4.5), we conclude that

$$a_1(E_1) = a_1(F_1) + b_1,$$

where

$$\begin{aligned} b_1 &= \sum_j r(p_*(Q_j(E_1)(m)))\ell = \sum_j r(p_*(Q_j(E_1)))\ell \\ &= \sum_j \ell(r(E_1|_{P_{x_j}}) + r(E_1|_{P_{z_j}}) - r(F_j(E_1))). \end{aligned}$$

Since $r(F_j(E_1)) \leq r(E_1|_{P_{x_j}})$, we have

$$b_1 \geq \sum_j \ell(r(E_1|_{P_{z_j}})).$$

Similarly, $b_1 \geq \sum_j \ell(r(E_1|_{P_{x_j}}))$, and hence

$$b_1 \geq \ell \sum_j \max \{r(E_1|_{P_{x_j}}), r(E_1|_{P_{z_j}})\}.$$

Then $a_1(E_1) = a_1(F_1) + b_1 \geq a_1(F_1) + \ell \sum_j \max \{r(E_1|_{P_{x_j}}), (r(E_1|_{P_{z_j}}))\}$. By definition,

$$\begin{aligned} d(E_1) &= a_1(E_1) - r(E)a_1(\mathcal{O}_X) \\ &\geq a_1(F_1) + \ell \sum_j \max \{r(E_1|_{P_{x_j}}), (r(E_1|_{P_{z_j}}))\} - r(F_1)a_1(\mathcal{O}_X) \\ &= a_1(F_1) + \ell \sum_j \max \{r(E_1|_{P_{x_j}}), (r(E_1|_{P_{z_j}}))\} - r(F_1)a_1(\mathcal{O}_Z) - \ell gr(F_1) \\ &= d(F_1) + \ell \sum_j (\max \{r(E_1|_{P_{x_j}}), (r(E_1|_{P_{z_j}}))\} - r(F_1)) \\ &\geq d(F_1). \end{aligned}$$

Since $r(F_1) = r(E_1)$, the result follows. \square

Proposition 4.2. *If E is an H_Z -semistable (respectively, H_Z -stable) vector bundle on Z with (E, h) giving a vector bundle F on X , then F is H -semistable (respectively, H -stable).*

Proof. Let $F'_1 \subset F$ be a torsion free subsheaf. Then we have $(p^*F'_1/\text{Torsion}) \subset E$. Let E_1 be the saturation of $(p^*F'_1/\text{Torsion})$ in E . Then E/E_1 is torsionfree and E_1 gives a torsionfree subsheaf $F_1 \subset F$ such that $F'_1 \subset F_1$. Since the quotient F_1/F'_1 is a torsion sheaf, we have $\mu(F'_1) \leq \mu(F_1)$. By Lemma 4.1, $\mu(F_1) \leq \mu(E_1)$ so that $\mu(F'_1) \leq \mu(E_1)$. If E is semistable (respectively, stable), then $\mu(E_1) \leq \mu(E)$ (respectively, $\mu(E_1) < \mu(E)$) so that

$$\mu(F'_1) \leq \mu(E_1) \leq \mu(E) = \mu(F) \text{ (respectively, } \mu(F'_1) \leq \mu(E_1) < \mu(E) = \mu(F))$$

proving the proposition. \square

Theorem 4.3. *Let $M_{X,H}^s(r, c_1, c_2)$ be the moduli space of H -stable (slope stable) vector bundles of rank r and Chern classes c_1, c_2 on X . Let $\Delta(r, c_1, c_2) = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2)$. Let F_Z denote the general fiber of Z . If $\Delta(r, c_1, c_2) \gg 0$, then $M_{X,H}^s(r, c_1, c_2)$ is a non-empty, smooth, irreducible, rational, quasiprojective variety in the following cases (c_1, c_2 below denote the classes on Z which are pull-backs of the classes c_1, c_2 on X):*

- (1) $c_1.F_Z = 1$ or $r-1 \pmod{r}$.
- (2) $c_1.F_Z = r-2 \pmod{r}$ and $c_2 - c_1^2/2 - c_1.\omega_Z/2 - (r-1) = 0 \pmod{2}$.
- (3) $c_1.F_Z = 2 \pmod{r}$ and $c_2 + c_1.C_0 - c_1^2/2 + c_1.\omega_Z/2 + 1 = 0 \pmod{2}$.

Proof. In [CM], Costa and Miró-Roig construct explicitly generic H -stable vector bundles E of rank r and Chern classes c_1, c_2 on Z in the above cases. It is easy to see that in each of these cases, the restrictions of E to all fibers are isomorphic. Using the bijective correspondence between GPBs on Z and vector bundles on X , together with Proposition 4.2, we can construct (generic) H -stable vector bundles on X of rank r and Chern classes c_1, c_2 . Thus $M_{X,H}^s(r, c_1, c_2)$ is non-empty in all the cases. By Theorem 3.7, this moduli space is irreducible and smooth.

We now turn to the question of rationality of $M_{X,H}^s(r, c_1, c_2)$. We shall in fact show that whenever $M_{Z,H_Z}^s(r, c_1, c_2)$ is rational, the variety $M_{X,H}^s(r, c_1, c_2)$ is also rational. Since the rationality of $M_{Z,H_Z}^s(r, c_1, c_2)$ is known in the cases listed in the statement of the theorem [CM], this will prove the theorem.

The moduli space $M_{Z,H_Z}(r, c_1, c_2)$ is a geometric invariant theoretic (GIT) quotient of a suitable quot scheme Q^{ss} by $\mathrm{PGL}(N)$. Let Q^s and $M_{Z,H_Z}^s(r, c_1, c_2)$ denote the subschemes corresponding to stable vector bundles, then $M_{Z,H_Z}^s(r, c_1, c_2) = Q^s // \mathrm{PGL}(N)$. Let

$$\mathcal{U}_Q \longrightarrow Q^s \times Z$$

be the universal quotient vector bundle.

For simplicity of exposition, let us take $g = 1$ and write $x_1 = x$, $z_1 = z$. We have vector bundles $\mathcal{U}_Q|_{Q^s \times P_x}$ and $(id \times \tau)^*(\mathcal{U}_Q|_{Q^s \times P_z})$ on $Q^s \times P_x$. Consider the sheaf

$$\mathcal{H}_Q = \mathrm{Hom}(\mathcal{U}_Q|_{Q^s \times P_x}, (id \times \tau)^*(\mathcal{U}_Q|_{Q^s \times P_z})).$$

Since scalars (the isotropy) acts trivially on the sheaf \mathcal{H}_Q , it descends to a sheaf $\mathcal{H}_{M,x}$ on $M_{Z,H_Z}^s(r, c_1, c_2) \times P_x$. Let

$$\tilde{H}_x = R p_{M_Z}^* \mathcal{H}_{M,x}$$

denote the direct image of $\mathcal{H}_{M,x}$ on $M_{Z,H_Z}^s(r, c_1, c_2)$. There is a Zariski open subset $M' \subset M_{Z,H_Z}^s(r, c_1, c_2)$ such that $\tilde{H}_x|_{M'}$ is locally free (by semi-continuity theorem). Hence there is a Zariski open subset $M'' \subset M'$ such that

$$\tilde{H}_x|_{M''} \cong M'' \times \mathbb{C}^n.$$

Therefore, if $M_{Z,H_Z}^s(r, c_1, c_2)$ is rational, then M'' and hence the total space of $\tilde{H}_x|_{M''}$ is rational.

Note that the fiber of $\tilde{H}_x|_{M'}$ over $[E] \in M'$ corresponds to the vector space

$$\mathrm{Hom}(E|_{P_x}, \tau^*(E|_{P_z})).$$

Since $\mathrm{Hom}(E|_{P_x}, \tau^*(E|_{P_z})) \supset \mathrm{Iso}(E|_{P_x}, \tau^*(E|_{P_z}))$, there is a Zariski open subset \tilde{H}' of the total space of $\tilde{H}_x|_{M''}$ which corresponds to generalized parabolic bundles on Z . By Proposition 4.2 and Section 4.1, this \tilde{H}' is isomorphic to an open subset of $M_{X,H}(r, c_1, c_2)$. It now follows that $M_{X,H}(r, c_1, c_2)$ is rational.

In the case of $g > 1$, we take the sheaf

$$\mathcal{H}_M \longrightarrow M_{Z,H_Z}^s(r, c_1, c_2) \times \coprod_j P_{x_j} = \coprod_j (M_{Z,H_Z}^s(r, c_1, c_2) \times P_{x_j}) \quad (4.6)$$

to be the sheaf whose restriction to $M_{Z,H_Z}^s(r, c_1, c_2) \times P_{x_j}$ is \mathcal{H}_{M,x_j} . There is a Zariski open subset $S \subset M_{Z,H_Z}^s(r, c_1, c_2)$ such that the restriction of the direct image of \mathcal{H}_M to S is a vector bundle. The rest of the argument works as in the one node case. \square

5. VECTOR BUNDLES OVER A REAL RULED SURFACE

In this section, we study moduli of vector bundles over a real rational ruled surface. Our goal is to prove that the moduli space $M(r, c_1, c_2)$ for vector bundles over a real rational ruled surface is rational as a real variety.

5.1. The real rational ruled surface.

Let σ be an anti-holomorphic involution on $\mathbb{P}_{\mathbb{C}}^1$. The pair $(\mathbb{P}_{\mathbb{C}}^1, \sigma)$ defines a smooth projective curve C defined over \mathbb{R} . Let $(x_1, z_1), \dots, (x_g, z_g)$ be distinct pairs of points of $\mathbb{P}_{\mathbb{C}}^1$. Let Y be a complex nodal curve of genus $g(Y) = g$ obtained by identifying points x_j with z_j for each j such that the involution σ induces an anti-holomorphic involution σ_Y on Y . We clarify that σ_Y need not fix pointwise $\{x_j\}_{j=1}^g$ and $\{z_j\}_{j=1}^g$. Note that $\sigma_Y(x_i) \in \{x_j, z_j\}$ if and only if $\sigma_Y(z_i) \in \{x_j, z_j\}$. Let y_1, \dots, y_g denote the nodes of Y with x_j, z_j identified to y_j . The pair (Y, σ_Y) defines a projective curve $Y_{\mathbb{R}}$, of arithmetic genus g , defined over \mathbb{R} . We have

$$Y_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} = Y.$$

The normalization of $Y_{\mathbb{R}}$ is the curve C , let

$$\pi' : C \longrightarrow Y_{\mathbb{R}}$$

be the normalization map. If C does not have any real points, then all real points of C lie in $\{y_j\}_{j=1}^g$.

Let $\mathcal{E}_{\mathbb{R}}$ be a real algebraic vector bundle of rank two over $Y_{\mathbb{R}}$, and let $\mathcal{E} = \mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its base change to \mathbb{C} , which is a complex vector bundle over Y . The vector bundle $\mathcal{E}_{\mathbb{R}}$ is defined by a pair $(\mathcal{E}, \sigma_{\mathcal{E}})$. Then $\sigma_{\mathcal{E}}$ induces an anti-holomorphic involution σ_X on $X := \mathbb{P}(\mathcal{E})$. The pair (X, σ_X) defines the real ruled surface

$$X_{\mathbb{R}} := \mathbb{P}(\mathcal{E}_{\mathbb{R}}) \longrightarrow Y_{\mathbb{R}},$$

and one has $X_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} = X$. Note that the anti-holomorphic involution σ_X lifts the anti-holomorphic involution σ_Y of Y . Since σ_Y permutes the nodes of Y , the involution σ_X permutes the fibers $\{P_j\}_{j=1}^g$, so $\coprod_j P_j$ is σ_X -invariant, thus $\coprod_j P_j$ is a real variety.

We have $Z = \mathbb{P}(\pi^* \mathcal{E})$; let $\sigma_Z := p^* \sigma_X$ be the anti-holomorphic involution induced by $\pi^* \sigma_{\mathcal{E}}$. The pair (Z, σ_Z) defines a ruled surface over C , defined over \mathbb{R} . Let $P := \coprod_j P_{x_j}$. Since we can canonically identify P_j with P_{x_j} for each j , we see that σ_Z leaves P invariant. Let

$$\sigma_P := \sigma_Z|_P : P \longrightarrow P$$

be the restriction.

The anti-holomorphic involution σ_Z on Z produces an anti-holomorphic involution σ_M on $M_{Z, H_Z}^s(r, c_1, c_2)$ defined by

$$\sigma_M(E) = \sigma_Z^*(\overline{E}).$$

There is a sheaf \mathcal{H}_M on $M_{Z, H_Z}^s(r, c_1, c_2) \times P = \coprod_j (M_{Z, H_Z}^s(r, c_1, c_2) \times P_{x_j})$ (defined by (4.6) in the proof of Theorem 4.3). For a general vector bundle $E \in M_{Z, H_Z}^s(r, c_1, c_2)$, one has $E|_{P_{x_j}} \cong \tau_j^*(E|_{P_{z_j}})$ for all j , where τ_j is defined in (2.1). We choose a σ_M -invariant open subset $M' \subset M_{Z, H_Z}^s(r, c_1, c_2)$ such that $E \in M'$ satisfies the above condition and $\mathcal{H}_M|_{M' \times P}$ is locally free.

Lemma 5.1. *The vector bundle $\mathcal{H}_M|_{M' \times P}$ is a real vector bundle.*

Proof. We shall construct an anti-holomorphic involution σ_H on $\mathcal{H}_M|_{M' \times P}$ lifting $\sigma_M \times \sigma_P$. For $(E, v_j) \in M' \times P_{x_j}$, we have

$$(\sigma_M \times \sigma_P)(E, v_j) = (\sigma_Z^* \overline{E}, \sigma_P(v_j)) \in M' \times \sigma_P(P_{x_j})$$

and $\sigma_Z^* \overline{E}|_{\sigma_P(P_{x_j})} = \overline{E}|_{P_{x_j}}$. Write

$$E_1 = E|_{P_{x_j}}, \quad E_2 = \tau^*(E|_{P_{z_j}}).$$

One has $\mathcal{H}_M|_{E \times P_{x_j}} = \text{Hom}(E_1, E_2)$ and

$$\mathcal{H}_M|_{\sigma_M E \times \sigma_P(P_{x_j})} = \text{Hom}(\overline{E}|_{P_{x_j}}, \tau^*(\overline{E}|_{P_{z_j}})) = \text{Hom}(\overline{E}_1, \overline{E}_2).$$

Any linear homomorphism $f : E_1 \rightarrow E_2$ induces a linear homomorphism

$$\overline{f} : \overline{E}_1 \rightarrow \overline{E}_2$$

such that $\overline{\overline{f}} = f$. Hence there is a natural anti-holomorphic involution σ_H on $\mathcal{H}_M|_{M' \times P}$ which lifts $\sigma_M \times \sigma_P$ and

$$\sigma_H : \text{Hom}(E_1, E_2) \rightarrow \text{Hom}(\overline{E}_1, \overline{E}_2)$$

is defined by $f \mapsto \overline{f}$. Since $\overline{\overline{f}} = f$, it follows that $\sigma_H^2 = \text{Id}$. Hence $\mathcal{H}_M|_{M' \times P}$ is a real vector bundle. \square

Theorem 5.2. *The variety $M_{X,H}(r, c_1, c_2)$ is rational as a real variety if $M_{Z,H_Z}(r, c_1, c_2)$ is rational as a real variety.*

Proof. Since $\mathcal{H}_M|_{M' \times P}$ is a real vector bundle (see Lemma 4.1), so is its direct image on M' . Let

$$V = p_{M'}^*(\mathcal{H}_M|_{M' \times P}).$$

The involution σ_H induces an anti-holomorphic involution σ_V on V . By replacing M' by a σ_M -invariant open subset if necessary, we see that there is a Zariski open subset U of the total space of V such that for $E \in M'$ the fiber $U_E = \bigoplus_j \text{Iso}(E_1, E_2)$ and $U \rightarrow M'$ is a locally trivial fiber bundle with fibers isomorphic to an (fixed) affine space. The involution σ_V keeps U invariant. Thus U is a real variety.

Since M' is rational as a real variety by our assumption, the above subset U is also rational as a real variety. The real variety $M_{X,H}(r, c_1, c_2)$ has an open subset isomorphic to U . It follows that $M_{X,H}(r, c_1, c_2)$ is rational as a real variety. \square

Theorem 5.3. *Let σ be an anti-holomorphic involution on $\mathbb{P}_{\mathbb{C}}^1$ defined by*

$$\sigma(x : y) = (\overline{y}, -\overline{x}).$$

The pair $(\mathbb{P}_{\mathbb{C}}^1, \sigma)$ defines a non-degenerate anisotropic conic C over \mathbb{R} .

Let $c_1 = C_0 + dF_Z, F_Z$ being the general fiber of Z . Let c_2, α, λ and m be integers satisfying

$$c_2 = \lambda(r-1) + \alpha, \quad 0 < \alpha \leq r-1, \quad m = d - c_2 - 1 - \lambda.$$

Assume that

$$\Delta(r, c_1, c_2) := \frac{1}{r}(c-2 - \frac{r-1}{2r}c_1^2) \gg 0.$$

Then $M_{X,H}(r, c_1, c_2)$ is rational as a real variety if and only if one of the following conditions holds:

(1) Both the integers m and $r - 1 - \alpha$ are even.

(2) The integer m is odd and α is even.

Proof. The theorem follows from Theorem 5.2 and the main theorem of [BS]. □

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